Stabilizing a Rotary Inverted Pendulum Based on Lyapunov Stability Theorem

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Outline

The Rotary Inverted Pendulum (RIP), is a well-known test platform to verify a control theory

- static instability.
- significant real-life applications.

For the swing-up and stabilizing control of the RIP, a variety of control methods had been applied :

- Proportional-Integral-Derivative (PID)
- Linear Quadratic Regulator (LQR)
- Particle Swarm Optimization (PSO)
- Genetic Algorithms (GA)
- Ant Colony Optimization (ACO)
- Fuzzy logic control
- H-∞ control
- Sliding mode control

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Lyapunov control method is a simple method of designing controls :

- Energy function can be selected as Lyapunov function and used Lyapunov direct method to stabilize RIP[1] [2].
  - Energy function considered the physical system and was constructed from physical standpoint.
  - Application is confined to the considered physical system.

- Instead of energy function, a logarithmic function is constructed as Lyapunov function and the controls are designed to stabilize RIP by Lyapunov method.
  - Construct Lyapunov function from mathematical standpoint.
  - Applicable to not only RIP but also the other physical systems.

[1] C. Aguilar-Ibanez, M. Suárez-Castanón and O. Gutiérres-Frias, The direct Lyapunov method for the stabilization of the Furuta pendulum, International Journal of Control, Vol.83, No.11, 2285-2293, 2010.

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The RIP is composed of a rotating arm driven by a motor, and a pendulum mounted on arm's rim, whose structure is shown in Figure 1.

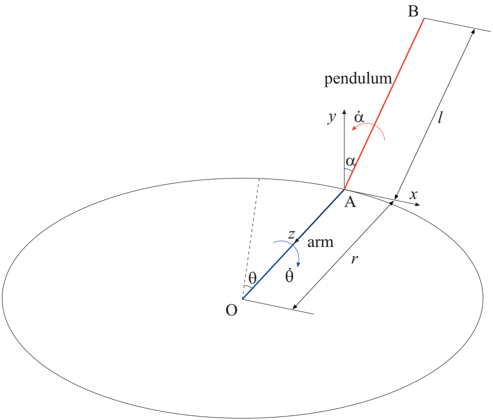

Figure 1: Schematic Diagram of RIP

By applying Newton method or Lagrange method, one can get the nonlinear mathematical model of RIP :

$$\begin{cases} \left(J_{eq}+mr^2\right)\ddot{\theta}+mlr\sin(\alpha)\dot{\alpha}^2-mlr\cos(\alpha)\ddot{\alpha}=T-B_q\dot{\theta} \\ \frac{4}{3}ml^2\ddot{\alpha}-mlr\cos(\alpha)\ddot{\theta}-mgl\sin(\alpha)=0 \end{cases} \tag{1}$$

where, V is voltage, T is the torque and given as

$$T=\eta_m\eta_g K_t K_g \frac{V-K_g K_m\dot{\theta}}{R_m} \tag{2}$$

The parameters in (1) and (2) are described as the Table.

Parameter	Description
J_{eq}	Moment of inertia at the load
m	Mass of pendulum arm
r	Rotating arm length
l	Length to pendulum?s center of mass
g	Gravitational constant
B_q	Viscous damping coefficient
K_t	Motor torque constant
K_g	System gear ratio
K_m	Back-EMF constant
R_m	Armature resistance
η_m	Motor efficiency
η_g	Gear efficiency

For small θ and α, $\cos(\alpha) \approx 1$ and $\sin(\alpha) \approx \alpha$. Placing the approximate expressions into (1) and solving (1), the state space model of RIP can be written as

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{3}$$

where $x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^T$, u is input voltage V, while

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bd}{E} & -\frac{cG}{E} & 0 \\ 0 & \frac{ad}{E} & -\frac{bG}{E} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ c\frac{\eta_m \eta_g K_t K_g}{R_m E} \\ b\frac{\eta_m \eta_g K_t K_g}{R_m E} \end{bmatrix}$$
$$C = diag(1,1,1,1), D = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$a = J_{eq} + mr^2, b = mlr, c = \frac{4}{3}ml^2, d = mgl, G = \frac{\eta_m \eta_g K_t K_m K_g^2 + BR_m}{R_m},$$
$$E = ac - b^2$$

Obviously, vertically upward position ($\alpha = 0$) and vertically downward position ($\alpha = \pi$) of pendulum are all equilibrium position for a fixed θ, i.e. the system (3) has more than one equilibrium states. In order to solve this problem, the status feedback technique is used to make the system (3) only one equilibrium state ($\alpha = 0, \theta = 0$).

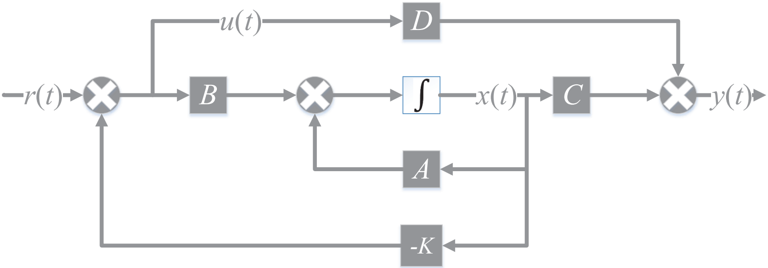

Figure 2: Block Diagram of Status Feedback

If the feedback control is noted as r, then (3) becomes

$$\begin{cases} \dot{x} = (A - BK)x + Br \\ \quad = \tilde{A}x + Br \\ y = (C - DK)x + Dr \\ \quad = \tilde{C}x + Dr \end{cases} \tag{4}$$

where, K is the gain vector of state feedback and $r = u + Kx = V + Kx$.

The aim of this paper is to design feedback control r (i.e. u) based on Lyapunov stability theorem so that the RIP can be stabilized at the position $\alpha = 0, \theta = 0$ from arbitrary initial position.

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Outline

The motivations of using Lyapunov method are:

- The procedure of designing controls is simple.
- The analytic expression of controls can be obtained.
- The controls designed by Lyapunov method can guarantee the system stability.

Based on Lyapunov stability theorem, the controls can be designed by Lyapunov method as follow:

1. Construct a function $V(x,t)$ which satisfies the conditions of Lyapunov function.
2. Calculate the first derivative of $V(x,t)$ for time $\dot{V}(x,t)$.
3. Design controls to make $\dot{V}(x,t) \leq 0$ and '$=$' holds if and only if $x = x_e$.

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Usually, quadratic function V_q is selected as Lyapunov function, i.e.

$$V_q(x,t) = x^T P x \tag{5}$$

where, P is non-negative matrix.
In order to obtain higher numerical accuracy and faster convergence speed, a logarithmic function V_l from quadratic function is selected as Lyapunov function, i.e.

$$V_l(x,t) = \ln\left(1 + x^T P x\right) \tag{6}$$

Numerical value between V_l and V_q

The Taylor expansion of $V_l(x,t)$ is

$$\begin{aligned} V_l(x,t) &= \ln\left(1 + x^T P x\right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(x^T P x\right)^n \\ &= x^T P x - \frac{1}{2}\left(x^T P x\right)^2 + \frac{1}{3}\left(x^T P x\right)^3 - \cdots \end{aligned}$$

from which on cen see that $V_q(x,t)$ is the first item of the Taylor expansion of $V_l(x,t)$, while $V_l(x,t)$ contains quadratic term, cubic term and higher order terms of $x^T P x$ relative to $V_q(x,t)$. Thus, $V_l(x,t)$ has a higher numerical accuracy than $V_q(x,t)$.

Convergence speed between V_l and V_q

Let $X = x^T P x \geq 0$, then

$$\frac{V_q}{V_l} = \frac{X}{\ln(1+X)}, \quad \frac{\dot{V}_q}{\dot{V}_l} = \frac{1}{\xi} = 1 + X$$

Construct a function as

$$f_1(X) = \frac{V_q}{\dot{V}_q} - \frac{V_l}{\dot{V}_l} = \frac{X}{(1+X)\dot{V}_l} - \frac{\ln(1+X)}{\dot{V}_l} = \frac{1}{\dot{V}_l}\tilde{f}_1(X) \qquad (7)$$

If $f_2(\Gamma) = \tilde{f}_1(X) = \frac{\Gamma-1}{\Gamma} - \ln(\Gamma)$ with $\Gamma = 1 + X \geq 1$, then

$$\dot{f}_2(\Gamma) = -\frac{\Gamma-1}{\Gamma^2} < 0$$

which means $f_2(\Gamma)$ is a monotone decreasing function, and $\max(f_2(\Gamma)) = f_2(1) = 0 = \max\left(\tilde{f}_1(0)\right)$ thereby,

$$\tilde{f}_1(X) \leq \max\left(\tilde{f}_1(X)\right) = \tilde{f}_1(0) = 0$$

Convergence speed between V_l and V_q

V_l is the constructed Lyapunov function, so $\dot{V}_l(x,t) \leq 0$ for $x \neq 0$ through designing controls. Taking into account (7), one can get that

$$f_1(\mathrm{X}) > 0 \ \ for \ \ \mathrm{X} \neq 0$$

which means $\frac{V_q}{\dot{V}_q} > \frac{V_l}{\dot{V}_l}$ for $x \neq 0$, namely V_l has faster convergence speed than V_q.

Through the above analysis, V_l has higher numerical accuracy and faster convergence speed than V_q, so the controls of stabilizing RIP are designed based on V_l.

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Outline

In order to design the controls under the condition of $\dot{V}_l(x,t) \le 0$ at any time, we need calculate the first order time derivative of $V_l(x,t)$

$$\begin{aligned}\dot{V}_l(x,t) &= \frac{1}{1+x^TPx}\left(\dot{x}^TPx + x^TP\dot{x}\right) \\ &= \frac{1}{1+x^TPx}\begin{pmatrix} x^T\left(\tilde{A}^TP+P\tilde{A}\right)x \\ +\left(B^TPx+x^TPB\right)r \end{pmatrix} \\ &= \frac{1}{1+x^TPx}\left(x^T\left(\tilde{A}^TP+P\tilde{A}\right)x+2x^TPBr\right) \\ &= \xi M + \xi N r\end{aligned} \tag{8}$$

where, $M = x^T\left(\tilde{A}^TP+P\tilde{A}\right)x$, $N = 2x^TPB$.

To ensure $\dot{V}_l(x,t) \le 0$, the feedback control is designed as

$$r = r_l = -\frac{M}{N} - k\cdot\xi\cdot N, k > 0 \tag{9}$$

Placing (9) into (8), it is easily get that

$$\dot{V}_l(x,t) = -k(\xi N)^2 \le 0 \ \ for \ \ x \ne 0$$

Similarly, the first order time derivative of $V_q(x,t)$ is

$$\begin{aligned}\dot{V}_q(x,t) &= \dot{x}^T P x + x^T P \dot{x} \\ &= M + Nu\end{aligned} \tag{10}$$

and the control is designed as

$$r = r_q = -\frac{M}{N} - kN, k > 0 \tag{11}$$

so that $\dot{V}_q(x,t) = -kN^2 \leq 0$ for $x \neq 0$.

Comparing (9) and (11), the first items of r_l and r_q are the same while the second item of r_l contain parameter ξ and r_q doesn't contain ξ.

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Two experiments are used to investigate the control r_l :

- The initial position of RIP is set to $x_0 = \left[\begin{array}{cccc} 1 & \pi & 1 & 0 \end{array} \right]^T$, which verifies that r_l can make the pendulum from vertically downward position to vertically upward position.
- The initial position of RIP is set to $x_0 = \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \end{array} \right]^T$, and r_l and r_q are used to stabilize RIP, respectively, then the control performances of r_l and r_q are compared.

Outline

1. Introduction
2. Mathematical Model of RIP
3. Design of Controls
   - Construction of Lyapunov Function
   - Design of Controls
4. **Experiments**
   - **System Parameter Settings**
   - Stability Control for Vertically Downward Initial State
   - Comparison of Controls r_l and r_q
5. Conclusions

The values of the parameters are given in Table 1.

Table 1: Values of Physical Parameters

J_{eq}	0.0033	m	0.125	r	0.215
l	0.1675	g	9.81	B_q	0.0040
K_t	0.0077	K_g	70	K_m	0.0077
R_m	2.6	η_m	0.69	η_g	0.90

Substituting the values into A and B in (3), then the eigenvalues of A are $\{0, -17.8671, 7.5266, -4.9783\}$ which means A is singular and the system (3) is unstable with more than one equilibrium states due to zero eigenvalue and positive eigenvalues.

The rank of $U_c = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$ is 4, which means the system is controllable. By applying state feedback, the system is controllable, stable and has only one equilibrium state.

The key of state feedback is to calculate the gain vector or matrix K. In this section, the desired closed-loop poles are set to $\{-1,-2,-3+3i,-3-3i\}$, then the closed-loop characteristic equation is

$$\begin{aligned} f(s) &= (s+1)(s+2)(s+3-3i)(s+3+3i) \\ &= s^4+9s^3+38s^2+66s+36 \end{aligned}$$

so

$$f(A) = A^4+9A^3+38A^2+66A+36I$$

Then, the gain vector K can be calculated as

$$\begin{aligned} K &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} U_c^{-1} f(A) \\ &= \begin{bmatrix} -0.0307 & 4.7360 & -0.6264 & 0.4086 \end{bmatrix} \end{aligned} \quad (12)$$

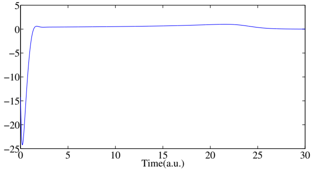
Outline

The initial state of RIP is set to $x_0 = \begin{bmatrix} 1 & \pi & 1 & 0 \end{bmatrix}^T$, while the final state is $x_f = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$. The control r_l in (9) is used to achieve this control task, in which the parameters are

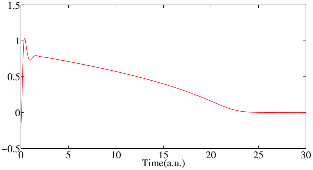
$$k = 0.1,$$
$$P = \begin{bmatrix} 1.6090 & -8.5878 & 0.7694 & -1.4376 \\ -8.5878 & 147.8866 & -11.3186 & 25.1413 \\ 0.7694 & -11.3186 & 0.9806 & -1.9023 \\ -1.4376 & 25.1413 & -1.9023 & 4.4263 \end{bmatrix} \quad (13)$$

where P is calculated by solving the Lyapunov equation $A^T P + PA = -I$.

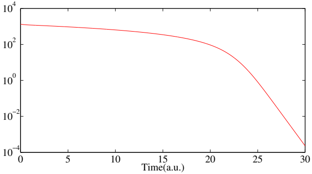
The experiment results are shown in Figure 3.



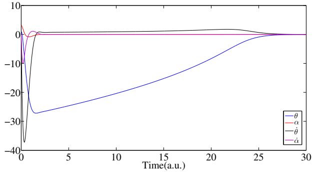
(a) Control $u = r_l - Kx$



(b) Feedback control r_l



(c) $V_q = x^T P x$

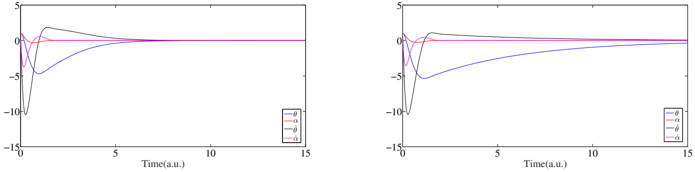


(d) System state x

Figure 3: Results of Stability Control for Vertically Downward Initial State

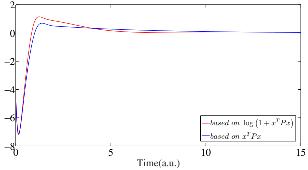
Outline

Let $x_0 = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^T$, $x_f = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$, k in r_l and r_q are 0.1 and 0.0009, respectively. The results are shown in Figure 4 and 5.



(a) System state x under u_l (b) System state x under u_q

Figure 4: Results of Comparative Experiments



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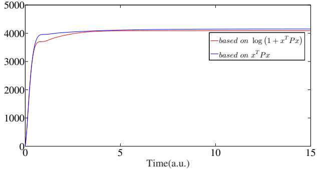
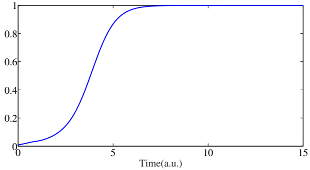
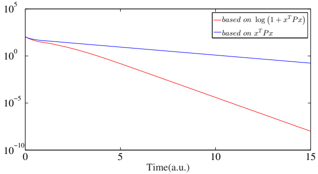


(b) Energies of control u



(c) Parameter ξ in control r_l



(d) $V_q = x^T Px$

Figure 5: Results of Comparative Experiments (cont.)

- The logarithmic function has higher numerical accuracy and faster convergence speed than quadratic function.
- The control designed can also achieve the swing-up control of RIP.
- Further works
  - Lyapunov control method is applicable to nonlinear system, so the nonlinear mathematical model of RIP can be considered as the controlled system, and the stabilizing control laws are designed by Lyapunov method.
  - The robustness analyses in theory should be researched.
  - The constructed logarithmic function can be used in other control methods, such as the performance function in optimal control.

Thank you for your attention!

Q&A